Multi-type continuous-state branching processes

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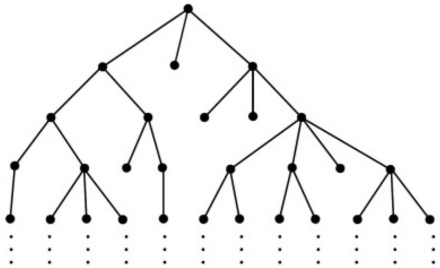
CB-PROCESSES.

CB-processes may be thought of as the continuous (in time and space) analogues of classical Galton-Watson processes.



CB-PROCESSES.

A continuous-state branching process (or CB-process) is a non-negative valued strong Markov process with probabilities $\mathbb{P}_x$ such that for any $x, y \geq 0$, $\mathbb{P}_{x+y}$ is equal in law to the convolution of $\mathbb{P}_x$ and $\mathbb{P}_y$, which is the branching property.

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In particular,

$$\mathbb{E}_x\left[e^{-\lambda X_t}\right] = \exp\{-x u_t(\lambda)\}, \qquad \text{for } \lambda \geq 0,$$

for some function $u_t(\lambda)$.

The function $u_t(\lambda)$ is determined by the integral equation

$$u_t(\lambda) = \lambda - \int_0^t \psi(u_s(\lambda))\mathrm{d}s$$

where ψ (branching mechanism of X) satisfies the Lévy-Khintchine formula

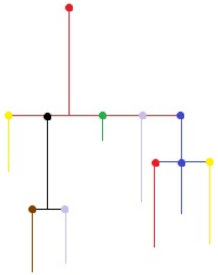
$$\psi(\lambda) = -a\lambda + \gamma^2\lambda^2 + \int_{(0,\infty)} \left(e^{-\lambda x} - 1 + \lambda x\right)\mu(\mathrm{d}x),$$

where $a \in \mathbb{R}$, $\gamma \geq 0$ and μ is a σ-finite measure such that

$$\int_{(0,\infty)} \left(x \wedge x^2\right)\mu(\mathrm{d}x) < \infty.$$

MULTI-TYPE CB-PROCESSES.

Multi-type Galton-Watson process



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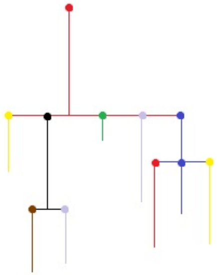

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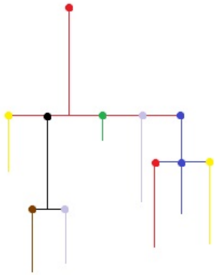

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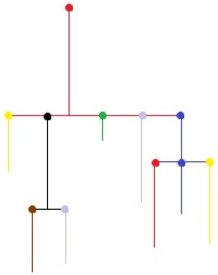

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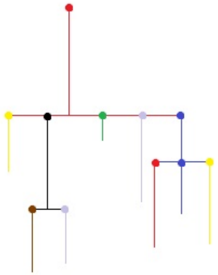

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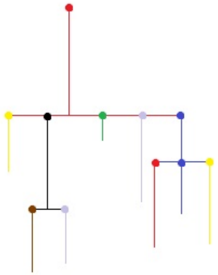

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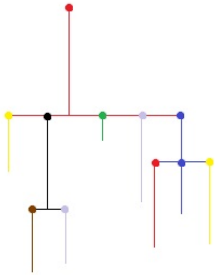

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- Continuous time.
- Vector in $[0, \infty)^{\mathbb{N}}$.
- Branching property
- with local and non-local branching mechanism.
- $$\langle f, \mu \rangle := \sum_{i \geq 1} f(i)\mu(i).$$

MULTI-TYPE CB-PROCESS

A multi-type continuous-state branching process is a $[0,\infty)^{\mathbb{N}}$-valued strong Markov process $X = (X_t : t \geq 0)$ with probabilities $\{\mathbf{P}_\mu, \mu \in \mathcal{M}(\mathbb{N})\}$ that satisfies the branching property:

$$\mathbf{E}_{\mu+\nu}[\mathrm{e}^{-\langle f, X_t \rangle}] = \mathbf{E}_\mu[\mathrm{e}^{-\langle f, X_t \rangle}]\mathbf{E}_\nu[\mathrm{e}^{-\langle f, X_t \rangle}].$$

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In particular,

$$\mathbf{E}_\mu[\mathrm{e}^{-\langle f, X_t\rangle}] = \exp\left\{-\langle V_t f, \mu\rangle\right\}, \quad \mu \in \mathcal{M}(\mathbb{N}),\ f \in \mathcal{B}^+(\mathbb{N}),$$

where, for $i \in \mathbb{N}$,

$$V_t f(i) = f(i) - \int_0^t \Big[\psi(i, V_s f(i)) + \phi(i, V_s f)\Big] \mathrm{d}s, \qquad t \geq 0.$$

Branching mechanisms

Local mechanism $\psi : \mathbb{N} \times [0, \infty) \to \mathbb{R}$.

$$\psi(i,z) = b(i)z + c(i)z^2 + \int_0^\infty (\mathrm{e}^{-zu} - 1 + zu)\ell(i, \mathrm{d}u), \qquad i \in \mathbb{N}, \quad z \geq 0,$$

where $b \in \mathcal{B}(\mathbb{N})$, $c \in \mathcal{B}^+(\mathbb{N})$ and, for each $i \in \mathbb{N}$, $(u \wedge u^2)\ell(i, \mathrm{d}u)$ is a bounded kernel from $\mathbb{N}$ to $(0, \infty)$.

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Non-local mechanism $\phi : \mathbb{N} \times \mathcal{B}^+(\mathbb{N}) \to \mathbb{R}$.

$$\phi(i,f) = -\beta(i)\left[d(i)\langle f, \pi_i\rangle + \int_0^\infty (1 - \mathrm{e}^{-u\langle f, \pi_i\rangle})\mathrm{n}(i, \mathrm{d}u)\right], \quad i \in \mathbb{N}, f \in \mathcal{B}^+(\mathbb{N})$$

where $d, \beta \in \mathcal{B}^+(\mathbb{N})$, π_i is a probability distribution on $\mathbb{N}\backslash\{i\}$ (specifically $\pi_i(i) = 0$, $i \in \mathbb{N}$) and, for $i \in \mathbb{N}$, $u\mathrm{n}(i, \mathrm{d}u)$ is a bounded kernel from $\mathbb{N}$.

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Intuitively, $X(i)$ evolves, in part from a local contribution which is that of a CB with mechanism $\psi(i, z)$, but also from a non-local contribution from other types. The mechanism $\phi(i, \cdot)$ dictates how this occurs. Each type $i \in \mathbb{N}$ seeds an infinitesimally small mass continuously at rate $\beta(i)d(i)\pi_i(j)$ on to sites $j \neq i$ (recall $\pi_i(i) = 0$, $i \in \mathbb{N}$). Moreover, it seeds an amount of mass $u > 0$ at rate $\beta(i)\mathrm{n}(i, \mathrm{d}u)$ to sites $j \neq i$ in proportion given by $\pi_i(j)$.

Extinction events

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Functional equation

Let define the vector $w(i) = -\log \mathbf{P}_{\delta_i}(\mathcal{E})$, $i \in \mathbb{N}$. Then w is a non-negative solution to

$$\psi(i, w(i)) + \phi(i, w) = 0, \qquad i \in \mathbb{N}. \tag{F. root}$$

LINEAR SEMIGROUP

Define the linear semigroup

$$M(t)_{ij} := \mathbf{E}_{\delta_i}[X_t(j)], \qquad t \geq 0,$$

and suppose that M is irreducible. (for any $i, j \in \mathbb{N}$, there exists $t > 0$ such that $M_{ij}(t) > 0$). Let

$$\Lambda = \sup \left\{ \lambda \geq -\infty : \int_0^\infty \mathrm{e}^{\lambda t} M(t)_{ij} \mathrm{d}t < \infty \right\},$$

be the spectral radius of M.

Local extinction dichotomy

Fix $\mu \in \mathcal{M}(\mathbb{N})$ such that $\sup\{n : \mu(n) > 0\} < \infty$. Moreover, suppose that

$$\sup_{i\in\mathbb{N}} \int_1^\infty (x \log x)\ell(i, \mathrm{d}x) + \sup_{i\in\mathbb{N}} \int_1^\infty (x \log x)\mathrm{n}(i, \mathrm{d}x) < \infty, \qquad \text{(xlogx)}$$

holds.

(i) For any finite number of states $A \subseteq \mathbb{N}$, $\mathbf{P}_\mu(\mathcal{L}_A) = 1$ if and only if $\Lambda \geq 0$.

(ii) The vector $v_A(i) = -\log \mathbf{P}_{\delta_i}(\mathcal{L}_A)$, $i \in \mathbb{N}$ is a solution for

$$\psi(i, v_A(i)) + \phi(i, v_A) = 0, \qquad i \in \mathbb{N}.$$

Open problems

1. By analogy with GWP, under (xlogx) condition we would expect that if $\Lambda < 0$, the value $-\Lambda$ would characterized the growth rate of any other type.

Conjecture $\exp\{\Lambda t\} X_i(t)$ converges a.s. to a non-trivial limit W_i as $t \to \infty$.

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If the number of types is infinite maybe this is not the case. So, an interesting question is how are the global and local growth rates related with each other?

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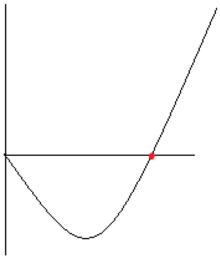
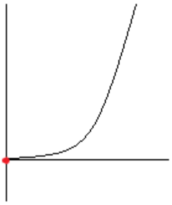
Global extinction versus global absorption. In the classical CB-process theory, absorption is the event defined as $\{\text{exist } t > 0 : X_t = 0\}$. Let denote by p_x and q_x the extinction and absorption probabilities started at x. If $\psi(\infty) = \infty$, then $p_x = \exp\{-x\eta\}$, where η is the biggest root of ψ.





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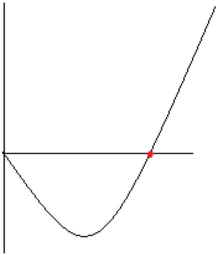

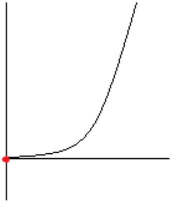

Moreover

$$q_x = p_x \mathbf{1}_{\{\int^\infty \psi(u)^{-1} du < \infty\}}.$$

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   b) Does it occur when mass disappears at finite time (absorption)?

5. What happens with the local extinction event? The irreducibility property will ensure that the mass in all the states will experience the same behaviour (all will be as in case a) or all will be as in case b)). Relations between local absorption and global absorption?

Open question

6. What would it change if we don't have irreducibility? What would we need to add in order to development this theory? If the matrix restricted to a communication class is positive definite, we could obtain its spectral radius. How is the asymptotic behaviour of the process?