Random Spectra

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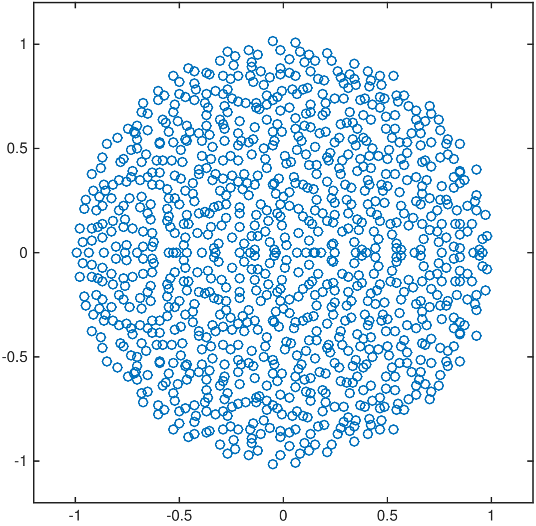
More precisely, I am interested in the **spectral density**

$$\varrho(\lambda; A) = \frac{1}{N} \sum_{i=1}^{N} \delta\left(\lambda - \lambda_i\right) \, ,$$

where $\lambda_1 \ldots \lambda_N$ are the eigenvalues of A.


1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1

The Circular Law

Let $\{A_N\}$ be a sequence of $N \times N$ random matrices with IID entries of zero mean and variance $1/N$. Then as $N \to \infty$ the spectral densities $\varrho(\lambda; A_N)$ converge strongly to the uniform density on the unit disc.

[Tao and Vu, 2008]

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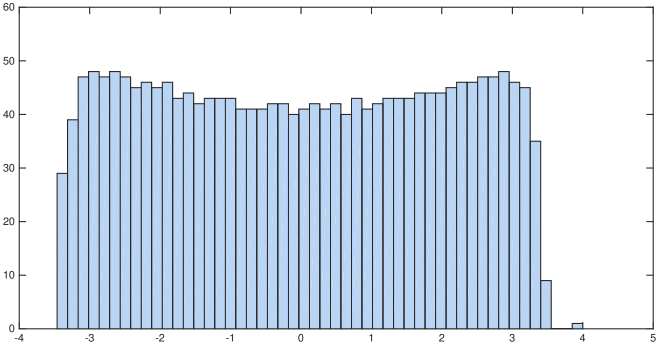
For example, let G be a random graph on N vertices in which every vertex has degree 4, and let A be it's adjacency matrix, i.e.

$$A_{ij} = \begin{cases} 1 & \text{if vertices } i \text{ and } j \text{ are neighbours} \\ 0 & \text{otherwise.} \end{cases}$$

How does the spectral density look now?


60
50
40
30
20
10
0
-4
-3
-2
-1
0
1
2
3
4
5

The Kesten-McKay Law

Let A_N be the adjacency matrix of a random k-regular graph on N vertices (i.e. an undirected simple graph in which every vertex has degree k, sampled uniformly at random from the collection of all such graphs). Then as $N \to \infty$

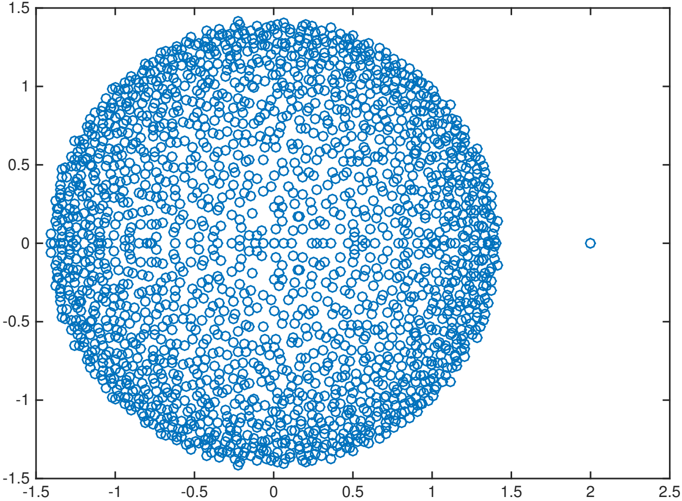
$$\varrho(\lambda; A_N) \to \frac{k\sqrt{4(k-1) - \lambda^2}}{2\pi(k^2 - \lambda^2)}\,.$$

What does this have to do with trees?

- In the limit of large N, the local neighbourhood of any given vertex in a k-regular random graph converges to a tree.
- One route to prove the Kesten-McKay law is to go via a technique known as the **cavity method** which maps the problem onto a typed branching process.

Open problem 1:

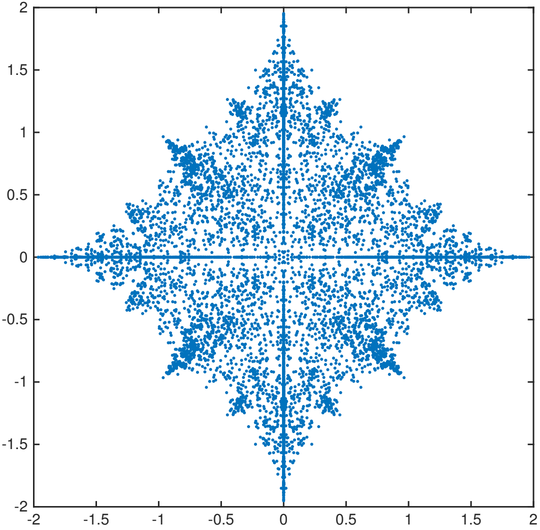
What is the limiting spectral density of "k-in k-out" random directed graphs? (A is not symmetric, but has exactly k non-zero entries in each row and column)



1.5
1
0.5
0
-0.5
-1
-1.5
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5

Open problem 2:

Consider random matrices where the upper and lower diagonal elements are chosen to be plus or minus one with equal probability (and the same for the top right and bottom left entries) with all other entries zero. What is the limiting spectrum?



2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2