Convergence of weighted trees

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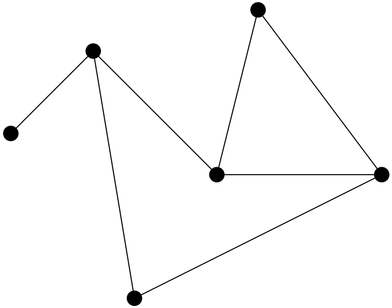
May 2016

Ising model

Let $G = (V, E)$ be a graph. For $\beta \geq 0$, define a measure $\mathbf{P}_{\beta,G}$ on $\{-1, 1\}^V$ by setting

$$\mathbf{P}_{\beta,G}(\sigma) = \frac{1}{Z} \exp\left\{\beta \sum_{v \sim w} \sigma_v \sigma_w\right\}.$$



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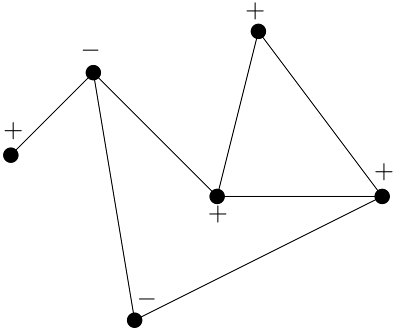

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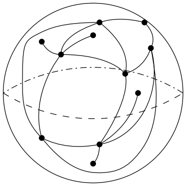
- $\beta = 0$: uniform pick
- $\beta \uparrow \infty$: pick $+$ or $-$ with pba $1/2$ and give all the vertices same spin

Big problem

- For each n, discretise the sphere $\mathbb{S}^2$ to get a finite set $\mathcal{M}_n$ (for example quadrangulations)



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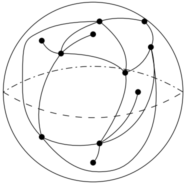

- For $G \in \mathcal{M}_n$, $\sigma \in \{-1, 1\}^V$:

$$\mathbb{P}_\beta((\mathbf{m}_n, \sigma_n) = (G, \sigma)) = \frac{1}{Z} \mathbf{P}_{\beta,G}(\sigma)$$

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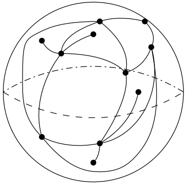

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- Find a scaling limit of $\mathbf{m}_n$.

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Theorem (Miermont (2013), Le Gall (2013))

When $\beta = 0$ (picking uniformly), for a large class of discritisations

$$\lim_{n\to\infty} \left(\mathbf{m}_n, \frac{1}{n^{1/4}} d_{gr} \right) = (\mathbf{m}_*, d)$$

in distribution under the Gromov-Hausdorff topology. The limiting space $(\mathbf{m}_*, d)$ is called the Brownian map.

Proposed problem

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Let $\mathcal{T}_n$ be the set of labelled trees with n vertices and pick $(T_n^{(\beta)}, \sigma_n)$ by

$$\mathbb{P}_\beta((T_n^{(\beta)}, \sigma_n) = (T, \sigma)) = \frac{1}{Z} \mathbf{P}_{\beta, T_n^{(\beta)}}(\sigma)$$

(possibly with boundary conditions).

Question

Does there exists a space $(T^{(\beta)}, d)$ and a $\gamma > 0$ such that

$$\lim_{n \to \infty} (T_n^{(\beta)}, n^{-\gamma} d_{gr}) = (T^{(\beta)}, d)$$

in distribution?

What is known?

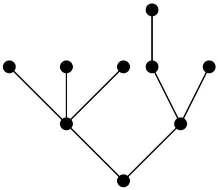
Theorem (Aldous (1991))

When $\beta = 0$ (picking uniformly),

$$\lim_{n\to\infty}\left(T_n^{(0)}, \frac{1}{n^{1/2}} d_{gr}\right) = (\mathbf{T}, \mathbf{d})$$

in distribution under the Gromov-Hausdorff topology. The limiting space $(\mathbf{T}, \mathbf{d})$ is called the continuum random tree.

An approach is to take an exploration of the tree



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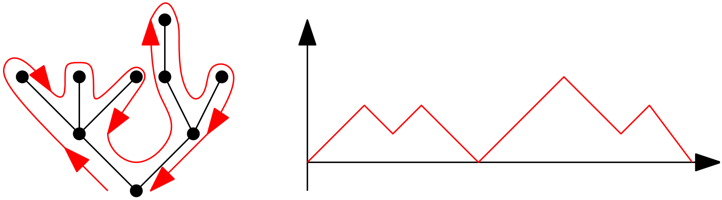

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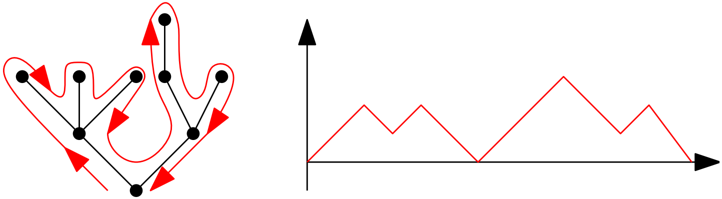

and try to incorporate the Ising model when exploring.

Thank You!